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The effects of trailing vorticity on the flow through highly loaded cascades

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This paper presents a procedure whereby three-dimensional inviscid flow through a highly loaded turbomachinery cascade of lifting lines can be treated by methods corresponding to classical aerodynamic theory. In contrast to earlier linearized (thin airfoil) three-dimensional theory, the present study allows analysis of the flow corresponding to the large turning and/or large pressure ratios induced by practical rotors or stators. For the sake of simplicity, the present paper is limited to incompressible flow through a highly loaded rectilinear cascade and to the design problem, i.e. given blade loading. Formulae are derived for both the mean and the three-dimensional components of the flow; in particular, the velocities at the blades induced by the trailing vorticity associated with non-uniform blade circulation are determined.

1. Introduction

The purpose of this paper is to introduce a method which allows the application of the ideas of classical aerodynamics (e.g. wing theory) to three-dimensional inviscid flow through highly loaded turbomachinery cascades. In order to keep the presentation of this new theory as brief and clear as possible, we restrict the present work to the case of incompressible flow through a highly loaded, isolated, rectilinear cascade, moving or stationary, with uniform inlet conditions, and assume that the absolute velocity far upstream is purely axial. The cascade of blades is assumed to be encased, for the purposes of the present treatment, in a duct of infinite length and width with finite height l (figure 1).

The theory we propose here, as it has been developed so far, is limited to the 'design problem', i.e. the blade loading is assumed given.

The basic physical idea which we follow is that when the blade circulation is non-uniform along the span the vorticity trailing downstream of the blades should be thought of as being convected approximately by the mean flow, which includes the usually large swirl induced by the blade row itself. By 'mean' we

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indicate here an average of any quantity over y. This treatment of the convection of the trailing vorticity is in contrast to the standard assumption, used in strictly linearized† theory, analogous to thin airfoil theory, which describes the trailing vorticity as being convected by the unperturbed flow, i.e. the flow specified far upstream of the blade row. The latter linearizing assumption has been used in all previous analyses which have applied classical aerodynamic theory to the problem of the three-dimensional flow through turbomachinery cascades.

In the simplest version of the present theory, the blades are represented by an infinite set, along y, of regularly spaced lines of bound vorticity, each aligned vertically in the z direction, perpendicular to the incoming flow in absolute co-ordinates, and each having circulation $\Gamma(z)$ (figure 1). The blades have uniform spacing s along y. In the case of a 'rotor' the blades move in the y direction with speed U. Thus, for an isolated rotor, i.e., for a situation in which we may, for example, neglect rotor–stator interaction, the flow field is actually a function of

$$y_1 \equiv y - Ut, \tag{1.1}$$

i.e. it is steady in relative co-ordinates.

If the vorticity trailing from the blades is approximated as being convected downstream by the mean flow, we shall find that we must correspondingly restrict the circulation on the blades to the form

$$\Gamma(z) = \overline{\Gamma} + \delta\Gamma(z), \quad \delta\Gamma = O(\epsilon), \tag{1.2}$$

i.e. $\delta\Gamma/\overline{\Gamma} \ll 1$. Here $\overline{\Gamma}$ is a constant, and equal to the average value,‡ over the span of the blade, of Γ:

$$\overline{\Gamma} \equiv \frac{1}{l}\int_0^l dz\, \Gamma(z), \quad \overline{\delta\Gamma} \equiv 0.$$

Thus, in the case of a rotor, for example, we limit ourselves to the situation of a cascade which is designed to do almost constant work. For a stator ($U = 0$), we limit ourselves to the case of almost uniform turning, in the rectilinear case; in the corresponding annular case one would have, in the mean, nearly free-vortex flow downstream of the blade row. Condition (1.2) represents the main restriction on the proposed new theory, i.e. $\delta\Gamma/\overline{\Gamma}$ must be a small parameter if the present theory is to apply.

On the other hand, $\overline{\Gamma}$ can be a large constant, consistent with the large turning, and/or large changes in pressure across the blade row, associated with practical, highly loaded cascades.

† We refer here to the theories of this type as 'strictly' linear, since they are consistent only with very small blade loading.

‡ This and

$$\tan \overline{\alpha}_2 = \frac{1}{l}\int_0^l \frac{\Gamma/s - U}{V_{-\infty}}\, dz \equiv \frac{\overline{\Gamma}/s - U}{V_{-\infty}}$$

are the only cases in the present paper where an overbar represents a *spanwise* average. In all other cases, we adopt the notation that a bar represents an average over 'y', i.e. an average over several blade passages.

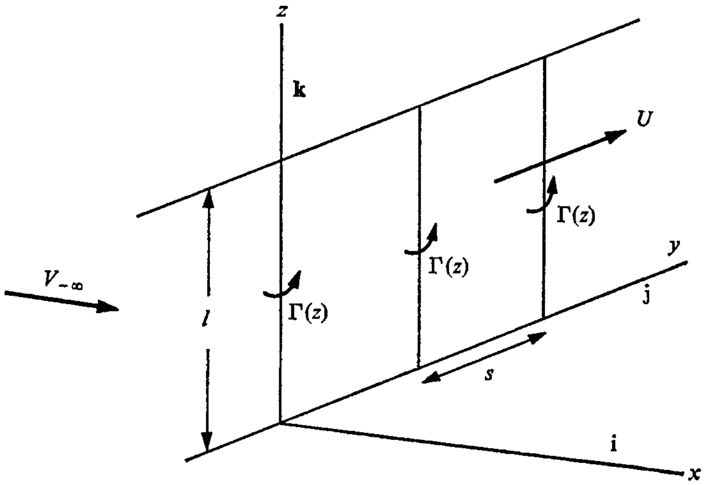

FIGURE 1. Notation.

The mean flow through the cascade, together with the three-dimensional perturbation of that flow, can be determined by the use of the above fundamental ideas. More significantly, we expect that the induced velocities associated with the trailing vorticity will be given more accurately by this approach than by the earlier, strictly linear, three-dimensional theories.

This paper is constructed as follows. In the next section, a brief review of the previous applications of three-dimensional, inviscid, classical aerodynamic theory to the flow through cascades is given.

In §3, a general approach to the nonlinear analysis of the inviscid incompressible equations of motion for the flow through the cascade in a special case of uniform inlet conditions is sketched. At this point our basic approximations are introduced and the general approach simplified. These approximations imply, as intended, that the trailing vorticity is indeed convected, very nearly, by the mean downstream flow. It is further argued that these approximations imply that the absolute flow field is correspondingly of the form

$$\mathbf{V} = \overline{\mathbf{V}} + \mathbf{v}, \quad \mathbf{v} \equiv (u, v, w) = O(\epsilon), \tag{1.3}$$

where $\mathbf{V}$ is the absolute velocity and $\overline{\mathbf{V}}$ its mean, usually $O(1)$. On the other hand, as indicated in (1.3), $\mathbf{v}$ is the three-dimensional flow field superimposed on $\overline{\mathbf{V}}$ and is 'small'; i.e. $|\mathbf{v}|/|\mathbf{V}| \ll 1$ within the present theory.

In §4, equations for the mean flow $\overline{\mathbf{V}}$, with special emphasis on the downstream flow, are derived using the actuator-disk approximation for simplicity. Nonlinear effects are included here. The mean downstream flow is matched with the mean upstream flow, through conditions imposed at the blade row, thus determining $\overline{\mathbf{V}}$ and related quantities completely. Of course, more accurate through-flow theories for the mean flow can be applied when appropriate.

In §5, equations for the three-dimensional flow field $\mathbf{v}$ are derived, and $\mathbf{v}$ is completely determined by applying appropriate three-dimensional matching conditions, imposed by the blade row, between the upstream and downstream flow. In particular, the velocities induced by the wakes of trailing vorticity are determined explicitly at the blades. At this stage advantage is taken of the fact that the shape of the surfaces of trailing vorticity can be directly calculated for given blade loading. The wakes become extremely distorted far downstream of the rotor but have a very simple form near the blades. As one might expect, this simple form is essentially determined by the mean outlet angle of the flow from the blades.

Finally, in § 6, an argument is presented which supports the use of a Trefftz-plane analysis for this problem. This opens up the possibility of computing the induced velocities associated with the trailing vorticity by an alternative method. When this analysis is carried out the answer supports the results of § 5. In both §§ 4 and 5 it is shown that the three-dimensional effects represented by the induced velocities are very strongly affected by the fact that the wakes leave the blades at very nearly the mean outlet angle from the blade row.

It has been shown (Morton 1974) that the extension of the present method to three-dimensional compressible flow through rectilinear cascades leads to similar results, with the addition of the expected propagating acoustic modes. Further, Cheng (1975) has successfully applied the new method to three-dimensional flow in annular cascades, for the incompressible case.

2. Earlier linearized theories: a review

The attempt to extend the analytical approach of classical aerodynamic theory to three-dimensional compressible flow through axial compressor blade rows, especially transonic rotors, began some time ago. One of the earliest attempts of this kind was that of McCune (1958*a*); this work led to a theory (McCune 1958*b*, *c*) valid for non-lifting rotors operating in the subsonic, transonic and supersonic regimes. This three-dimensional theory, as developed at that time, was a small perturbation analysis with two associated assumptions: first, it was assumed that the thickness problem could be treated separately from the lifting problem; and second, it was assumed that the blades were thin, thereby inducing only weak disturbances of the incoming flow.

Apart from the primary aim of obtaining a self-consistent theory of the three-dimensional flow through an annular cascade, a fundamental goal of the approach at that time was to establish to what extent two-dimensional cascade analysis (strip theory) could be derived from three-dimensional theory, thus attempting to establish the regimes of validity of applying two-dimensional cascade data, then in wide use, to the design of three-dimensional rotors and stators. It was found that there was no difficulty in justifying strip theory (and hence, presumably, the use of the corresponding cascade data) in the purely subsonic and supersonic cases. On the other hand, in the transonic regime, no relation could be established between two-dimensional cascade results and three-dimensional analysis. This fact left an obvious dilemma for all involved in this complex field of study.

It was not until 1967 that a corresponding three-dimensional study of a lifting rotor was published (Okurounmu 1967; Okurounmu & McCune 1970). In this theory the rotor was represented by B lines of bound vorticity of given strength $\Gamma(r)$ along the span, rotating at speed ω in an annular duct of infinite length. Again, the analysis was a strictly small perturbation theory, in the sense that it was explicitly assumed that the blade loading itself was 'small' ($\Gamma/V_{-\infty}s \ll 1$, where $V_{-\infty}$ is the inlet velocity, which is assumed to be purely axial, and uniform, while s is the blade-to-blade spacing). Thus, in this earlier theory, only slight turning of the flow and low pressure ratios could be analysed. However, it was once more found that the transonic case did not correspond to two-dimensional cascade theory (except for $\Gamma \equiv$ constant) and that there were in general very important differences between the predictions of the two approaches under comparable loading.

One of the features of the lifting theory was the inclusion of the wakes of trailing vorticity emanating from each blade, and the development of the ability to predict the velocity induced by this vorticity, at the blades. However, it was again assumed that the trailing vorticity was convected by the unperturbed flow, following the technique introduced by Reissner (1937).

A review of the three-dimensional strictly linear theory available in 1970 was presented by McCune & Okurounma (1974). It was pointed out by them that the tangential mean of the three-dimensional effects described in the existing theories corresponded to linearized axisymmetric through-flow analysis, with three-dimensional fluctuations superimposed, and it was suggested that this fact might lead to a way to relax some of the small perturbation assumptions. The present paper is to a certain extent a development of that suggestion.

The linearized analysis was also extended to second order in perturbation quantities in order to compute the 'losses' (Okurounmu & McCune 1970). These losses included those due to the wakes of shed vorticity (analogous to induced drag in wing theory) plus, in the transonic case, those due to the wave energy radiated away by the acoustic modes excited in that case. While the losses were small, for the linearized study, they scaled with $\bar{\Gamma}$, which indicated that they could become important for highly loaded blade rows.

The work on the lifting problem up to this point was limited to the 'design problem'; i.e. given the loading, find the blade shape, induced angles, etc. To overcome this limitation, a Prandtl-type lifting-line theory was developed (McCune & Dharwadkar 1972) for subsonic lifting rotors, allowing treatment of both the 'design' and 'off-design' problems. One of the important conclusions of that study is that strong deviations of the flow angles develop near the hub and tip sections whenever $\Gamma(r)$ is non-uniform along the span. It was also proved in that paper that no solution of the lifting problem is possible unless $d\Gamma/dr = 0$ at the hub and tip. This result had been pointed out earlier (Falcão 1970).

A lifting-surface theory of axial compressor blade rows was developed by Namba (1974), who went on to find a method of solving both the design and off-design problems, even in the transonic regime. He concluded, in agreement with earlier studies, that two-dimensional cascade analysis fails in the transonic regime, although it gives acceptable results in the subsonic and supersonic cases. Namba's study is linearized with respect to the far upstream flow.

Okurounmu & McCune (1971, 1974*a*, *b*) also developed a linearized lifting-surface theory for an axial-flow compressor and demonstrated the wide variation in blade shape required for given loading, showing that completely different blade shapes are required, in the transonic case, depending on whether or not $d\Gamma/dr = 0$. Their study is limited to the design problem.

As already emphasized, the above analyses allow only weak disturbance of the conditions specified far upstream, a fact which has restricted their usefulness in practical compressor design. The present paper is offered as a possible means of relaxing this limitation.

3. General approach and basic approximations

In this section we apply our proposed new method to the study of the three-dimensional flow through an isolated blade row in an infinite rectilinear duct, using the assumptions described in the introduction.

It is well known, for the rectilinear geometry of figure 1, that the circulation per blade is

$$\Gamma = \Delta \overline{V}_y / s, \tag{3.1}$$

where $\Delta \overline{V}_y$ is the tangential average of the change across the rotor in the y component of the absolute velocity [see footnote under (1.2)].

It is also well known that $\overline{V}_y = \overline{V}_y[\psi(x, z)]$ and hence, for our case, $\overline{V}_y = 0$ upstream of the blade row. Thus we also have $\Gamma = \Gamma[\psi(x,z)]$. Here, $\psi(x,z)$ is the stream function of the mean flow:

$$\overline{V}_x = \partial\psi/\partial z, \quad \overline{V}_z = -\partial\psi/\partial x, \tag{3.2}$$

where we have chosen Cartesian co-ordinates (x,y,z), with x the axial direction and z the spanwise (figure 1).

If the blade row is isolated, i.e. no significant interference is present between rotor, stator and/or guide vanes, then there is a co-ordinate system in which the flow is steady [see (1.1)]. For an isolated rotor, as we have stated, the flow is steady in the relative system with the result that $\partial/\partial t = -U\,\partial/\partial y$ in absolute co-ordinates. Using this result in the equations of motion in the absolute system, one finds that the equations of motion reduce to

$$\mathbf{W} \times \mathbf{\Omega} = \nabla(p/\rho + \tfrac{1}{2}V^2 - UV_y + \tfrac{1}{2}U^2) = 0 \tag{3.3}$$

both upstream and downstream of the rotor, where the last equality follows from Euler's turbine equation for incompressible flow and the assumed uniform inlet conditions, and where $\mathbf{\Omega} \equiv \operatorname{curl} \mathbf{V}$. Here $\mathbf{V}$ is the absolute velocity, already defined, while $\mathbf{W}$ is the relative velocity $\mathbf{W} = \mathbf{V} - U\mathbf{j}$ (figure 1) in the case of a moving blade row.

For the incompressible case, we have simply

$$\operatorname{div} \mathbf{V} = \operatorname{div} \mathbf{W} = 0, \quad \operatorname{div} \overline{\mathbf{V}} = 0. \qquad (3.4), (3.5)$$

Of course, (3.5) is automatically satisfied by (3.2).

For uniform upstream conditions, the solution of (3.3) in the upstream flow is simply $\mathbf{\Omega} = 0$, and we may determine, with appropriate boundary conditions, the form of the upstream flow by standard methods corresponding to potential-flow theory. However, the final determination of the upstream flow depends on matching it in some fashion with the downstream flow, through conditions imposed by the rotor. The downstream flow is considerably more complicated than the upstream flow, because of the vorticity trailing behind the blades. As already indicated, it is the method by which this trailing vorticity is treated which determines whether we must limit ourselves strictly to small perturbations of the incoming flow or whether large turning can be included in the theory.

The general solution of (3.3) is simply

$$\mathbf{\Omega} = \hat{\lambda}\mathbf{W}, \tag{3.6}$$

where $\hat{\lambda}$ is any scalar function. Imposing (3.4) on (3.6), we are led, however, to a condition on $\hat{\lambda}$, namely $\mathbf{W}\,.\,\nabla\hat{\lambda} = 0$. Now, since $\operatorname{div}\mathbf{W} = 0$, we may describe $\mathbf{W}$ quite generally by $\mathbf{W} = \nabla\alpha \times \nabla\beta$, where α and β are scalar functions of (x, y, z). As usual, the intersections of the surfaces $\alpha =$ constant and $\beta =$ constant are simply the streamlines of $\mathbf{W}$. Since we can describe $\mathbf{W}$ in this form, it is apparent that $\mathbf{W}\,.\,\nabla\hat{\lambda} = 0$ is automatically satisfied if

$$\hat{\lambda} = \hat{\lambda}(\alpha, \beta). \tag{3.7}$$

The physical meaning of (3.6) is that the downstream vorticity is oriented in the direction of and is convected along streamlines associated with $\mathbf{W}$, the exact relative velocity; also $\hat{\lambda}$ is constant along a streamline in view of (3.7).

Exact use of the general theory described above will not be attempted here. Rather, at this point we suggest approximations appropriate to our present purpose. For certain cases the vorticity can be described as being convected approximately by the mean relative flow $\overline{\mathbf{W}}$, rather than by $\mathbf{W}$ itself, as in (3.6). For this to be true we must expect $\mathbf{V}$ to be made up of a zeroth-order mean flow $\overline{\mathbf{V}}$ plus 'small' $[O(\epsilon)]$ three-dimensional perturbations. Thus [to repeat (1.3)], we must have

$$\begin{aligned} \mathbf{V} &= \overline{\mathbf{V}} + \mathbf{v}, \quad \mathbf{v} = O(\epsilon), \\ &= \overline{\mathbf{V}}(x, z) + \mathbf{v}(x, y, z). \end{aligned} \tag{3.8}$$

This condition requires also that the trailing vorticity be 'small', $O(\epsilon)$; that is, condition (1.2) must apply, and we have, correspondingly, the restriction

$$\Gamma'(\psi) = O(\epsilon). \tag{3.9}$$

Here the prime indicates a derivative with respect to the argument shown; as usual, the trailing vorticity is proportional to Γ'. On the other hand, there is no restriction on the magnitude of $\overline{\Gamma}$, and therefore the new theory holds, when so required, for large deflexion and large pressure changes across the blade row. For our present purposes, we simply require $V_{-\infty}s\Gamma'(\psi)/\overline{\Gamma} \ll 1$.

The assumptions introduced above allow us to simplify (3.6), and its consequence $\mathbf{W}\,.\,\nabla\hat{\lambda} = 0$, as follows:

$$\mathbf{\Omega} = \lambda\overline{\mathbf{W}} + O(\epsilon^2), \quad \overline{\mathbf{W}}\,.\,\nabla\lambda \doteq 0, \tag{3.10), (3.11}$$

where now $\overline{\mathbf{W}}$ is the mean relative velocity. Thus, as desired, (3.10) [see also (3.14)] implies the approximate convection of the vorticity by the mean relative flow. Further, $\overline{\mathbf{W}}$ can be described in a simpler way than can $\mathbf{W}$ itself; namely, since $\operatorname{div} \overline{\mathbf{W}} = 0$

$$\overline{\mathbf{W}} = \nabla\alpha \times \nabla\psi, \tag{3.12}$$

where now $\psi(x,z)$ is the stream function for the mean flow [see (3.2)]. Moreover, without loss of generality we may define

$$\alpha \equiv y - f(x,z). \tag{3.13}$$

The x and z components of (3.12), with α given by (3.13), simply reproduce (3.2). We show in §5 that the function $f(x,z)$ is determined from the y component of (3.12), for given blade loading, and α becomes a known quantity. Finally, (3.7) is replaced by

$$\lambda = \lambda(\alpha, \psi), \tag{3.14}$$

which automatically satisfies (3.11) in view of (3.12). The stream function $\psi(x,z)$ will be determined by the solution of the equations of the mean flow; these equations will be developed in §4. λ in (3.14) is constant along mean streamlines ψ = constant.

The vorticity in the downstream flow is that shed from the blades and is necessarily zero between the sheets of concentrated trailing vorticity. Hence the form of $\lambda(\alpha, \psi)$ can be expected to be

$$\lambda(\alpha, \psi) = F(\psi) \sum_{n=-\infty}^{n=\infty} \delta(\alpha - ns), \tag{3.15}$$

where

$$\sum_{n=-\infty}^{\infty} \delta(\alpha - ns)$$

is a periodic delta function with the argument shown. Inserting (3.15) in (3.10) and averaging over y (or α) leads to the equation for the mean downstream flow:

$$\bar{\mathbf{\Omega}} = (F(\psi)/s)\,\overline{\mathbf{W}}. \tag{3.16}$$

The x and z components of (3.16) can be used to show that

$$F(\psi) = -\Gamma'(\psi) \tag{3.17}$$

and the y component of (3.16), with (3.17), becomes the expected 'Beltrami' equation for the mean downstream flow. This equation thus yields the equation for $\psi(x,z)$, which can be solved for the mean downstream flow, given $\Gamma(\psi)$ or $\overline{V}_y(x,z)$.

The wakes of shed vorticity lie in the surfaces $\alpha = 0, \pm s, \pm 2s, \pm 3s, \ldots$, where α is determined in §5.

4. The mean flow

Because of the assumed uniform inlet conditions, and our neglect of viscous effects, the upstream flow is potential,

$$\mathbf{V}^u = \nabla\phi^u, \tag{4.1}$$

and with (3.4) this yields

$$\nabla^2\phi^u = 0. \tag{4.2}$$

This equation, of course, does not imply that the perturbations of the incoming flow are small. The boundary conditions are

$$W_z = V_z = 0 \quad \text{at} \quad z = 0, l \quad (\text{all } x) \tag{4.3}$$

and the corresponding solution of (4.2) is standard:

$$\phi^u = V_{-\infty} x + \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} A^u_{nm} \exp(\lambda_{nm} x) \cos(n\pi z/l) \exp(2\pi i m y/s), \tag{4.4}$$

where the real part is implied, and

$$\lambda^2_{nm} \equiv [(2\pi m/s)^2 + (n\pi/l)^2]. \tag{4.5}$$

As previously stated, $\bar{V}^u_y = 0$ in this case, and (3.1) is accordingly simplified.

The $m = 0$ component of ϕ^u (or $\mathbf{V}^u$) automatically gives the mean flow $\bar{\mathbf{V}}^u$ in the upstream region; we shall use this result, matched across the blade row with $\bar{\mathbf{V}}^d$ (determined below),† to obtain the coefficients A^u_{n0}, and the corresponding A^d_{n0}. The principal subject of the remainder of this section will be the determination of $\bar{\mathbf{V}}^d$, which includes the effects of the vorticity shed into the region downstream of the blade row.

The components of ϕ^u [equation (4.4)] with $m \neq 0$ represent the three-dimensional perturbations of the mean flow in the upstream region; these will be matched in §5 with the corresponding perturbation results for the downstream flow, thus determining the three-dimensional perturbations, both upstream and downstream, and enabling us to compute the induced velocities at the lifting lines representing the blades.

Using (3.2) and (3.1), together with the fact that $\bar{V}^d_y = \bar{V}^d_y(\psi) = \Gamma(\psi)/s$, for uniform axial inlet conditions, we find

$$\nabla^2 \psi = -\frac{\Gamma'(\psi)}{s}\left[\frac{\Gamma(\psi)}{s} - U\right], \tag{4.6}$$

where we have also used the definition of $\mathbf{W}$ given above (3.4). An exact solution of this equation for $\Gamma' =$ constant has been worked out in detail by Oates (1971), and pointed out much earlier by Bragg & Hawthorne (1950). However, Falcão's remark (1970) and the proof given by McCune & Dharwadkar (1972) that Γ' must equal zero at $z = 0, l$ lead us to reject this solution for the purposes of the present study. Indeed, if we do adopt Oates' tempting model, we later find, in studying the three-dimensional effects, infinite induced velocities at the ends of the blades; cf. Thwaites (1960, p. 277) for an analogy in classical wing theory.

Accordingly, we choose an alternative approach, suggested by the approximation (1.2) – or equivalently (3.9) – in which we make use of the fact that

$$\bar{V}^d_y = \frac{\bar{\Gamma}}{s} + \frac{1}{s}\delta\Gamma(\psi). \tag{4.7}$$

† We use the superscripts u and d to indicate upstream and downstream quantities, respectively. However, we drop the superscripts whenever they clearly are not needed.

It is well known (and will also be shown explicitly below) that, when (4.7) holds, it is also true, for incompressible flow, that $\psi^d = V_{-\infty}z + O(\delta\Gamma)$, where $V_{-\infty}$ is, as before, the far-upstream (purely axial) absolute velocity. In that case we can write

$$\Gamma'(\psi) = \frac{1}{V_{-\infty}}\frac{d\delta\Gamma}{dz} + O(\delta\Gamma)^2 \tag{4.8}$$

and equivalently

$$\delta\Gamma(\psi) = \delta\Gamma(z) + O(\delta\Gamma)^2. \tag{4.9}$$

Then (4.6) becomes, omitting terms of order ϵ^2,

$$\nabla^2\psi \cong -\frac{1}{V_{-\infty}s}\frac{d\delta\Gamma}{dz}\left(\frac{\overline{\Gamma}}{s} - U\right). \tag{4.10}$$

Consistent with our requirement that $\Gamma' = 0$ at $z = 0, l$, we introduce the expansion (recall that $\delta\Gamma$ is specified)

$$\delta\Gamma = \sum_{n=1}^{\infty} \Gamma_n \cos\frac{n\pi z}{l}, \tag{4.11}$$

whereupon ψ in (4.10) has the solution

$$\psi = V_{-\infty}z + \sum_{n=1}^{\infty} A^d_{n0}\exp\left(-n\pi x/l\right)\sin\frac{n\pi z}{l} - \sum_{n=1}^{\infty}\frac{\Gamma_n}{V_{-\infty}s}\left(\frac{\overline{\Gamma}}{s} - U\right)\left(\frac{l}{n\pi}\right)\sin\frac{n\pi z}{l}. \tag{4.12}$$

Of course, the Γ_n are automatically $O(\delta\Gamma)$, and it will turn out, upon our carrying out the matching across the blade row, that the A^d_{n0} are also $O(\delta\Gamma)$. Thus ψ is in fact of the form stated above (4.8).

For the mean flow, the most useful form of the matching conditions across the blade row are as follows.

(a) Continuity of mass across the blade row, namely

$$\left.\frac{\partial\psi}{\partial z}\right|_{x=0-} = \left.\frac{\partial\psi}{\partial z}\right|_{x=0+},$$

or

$$\left.\frac{\partial\overline{\phi}^u}{\partial x}\right|_{x=0-} = \left.\frac{\partial\psi}{\partial z}\right|_{x=0+}. \tag{4.13}$$

(b) Because there is no spanwise force on the blades, the jump across the blade row of the tangential mean of the spanwise component of the absolute velocity is zero; hence

$$\left.\frac{\partial\overline{\phi}^u}{\partial z}\right|_{x=0-} = -\left.\frac{\partial\psi}{\partial x}\right|_{x=0+}. \tag{4.14}$$

Requirements (b) and (a) lead immediately to the results that

$$A^u_{n0} = -A^d_{n0}$$

and

$$-A^u_{n0} = A^d_{n0} = \frac{\overline{\Gamma}/s - U}{2V_{-\infty}}\frac{\Gamma_n}{s}\left(\frac{l}{n\pi}\right) + O(\epsilon^2), \quad n \geqslant 1,$$

$$\equiv \tfrac{1}{2}\tan\overline{\alpha}_2\frac{\Gamma_n}{s}\left(\frac{l}{n\pi}\right) + O(\epsilon^2). \tag{4.15}$$

(A^u_{00} remains undetermined, but is not required in $\overline{\mathbf{V}}^u$ or in any other physical quantity.)

Notice that these results imply that

$$\bar{V}_x = V_{-\infty} + \frac{\bar{\Gamma}/s - U}{2V_{-\infty}} \sum_{n=1}^{\infty} \frac{\Gamma_n}{s} \exp(-n\pi x/l) \cos\frac{n\pi z}{l} - \frac{\bar{\Gamma}/s - U}{V_{-\infty}} (\delta\Gamma/s) \tag{4.16}$$

and hence $$\bar{V}_x(\infty) - V_{-\infty} = 2[\bar{V}_x(0) - V_{-\infty}] = \tan\bar{\alpha}_2[(\bar{\Gamma} - \Gamma)/s]. \tag{4.17}$$

Note further, from (4.13) and (4.14), that the jump across the blade row of the tangential mean of the absolute velocity is just

$$\Delta\bar{\mathbf{V}} = \Delta\bar{V}_y \mathbf{j} = \mathbf{j}\left(\frac{\Gamma(z)}{s}\right) = \mathbf{j}\left(\frac{\bar{\Gamma}}{s} + \frac{\delta\Gamma}{s}\right). \tag{4.18}$$

5. The perturbation flow: wakes and induced velocities

In view of (3.6) or (3.10), we write

$$\mathbf{V} = \nabla\phi + \mathbf{A} = \bar{\mathbf{V}} + \mathbf{v}. \tag{5.1}$$

We already know from § 4 that $\mathbf{A} = 0$ upstream of the blade row and, in fact, the perturbation part of the upstream flow is given directly by the $m \neq 0$ terms in (4.4), provided we are able to determine the coefficients A^u_{nm}. Our task then is to obtain $\mathbf{A}$ and ϕ in the downstream flow, separate out the mean parts (which are already known from the previous section), and develop an expression for $\mathbf{v}^d$. Finally, with appropriate matching of the perturbation terms across the blade row, the entire three-dimensional flow field satisfying the restrictions of the present treatment will be determined (for a given loading).

From (3.10), (3.12), (3.15) and (3.17) we have immediately

$$\begin{aligned} \operatorname{curl}\mathbf{A} &= \nabla\Gamma \times \nabla\left[\sum_{n=-\infty}^{\infty} H(\alpha - ns)\right] \\ &\equiv \nabla\Gamma \times \nabla H_n(\alpha), \end{aligned} \tag{5.2}$$

where $$\sum_{n=-\infty}^{\infty} H(\alpha - ns)$$

is a repeated Heaviside step function with positive unit steps at

$$\alpha = 0, \pm s, \pm 2s, \ldots$$

(the step at $\alpha = 0$ is from $-\frac{1}{2}$ to $+\frac{1}{2}$). Equation (5.2) follows from the fact that the periodic delta function appearing in $\lambda(\alpha, \psi)$ is the derivative with respect to α of

$$H_n(\alpha) \equiv \sum_{n=-\infty}^{\infty} H(\alpha - ns)$$

(Lighthill 1958). Thus

$$\nabla H_n(\alpha) = \frac{dH_n}{d\alpha}\nabla\alpha = \nabla\alpha \sum_{n=-\infty}^{\infty} \delta(\alpha - ns)$$

and similarly $$\nabla\Gamma = (d\Gamma/d\psi)\nabla\psi.$$

Equation (5.2) has the immediate integrals

$$\mathbf{A}_1 = -H_n(\alpha)\nabla\Gamma, \quad \mathbf{A}_2 = \Gamma\nabla H_n(\alpha), \tag{5.3}$$

the difference between the two being a curl-free vector. $H_n(\alpha)$ has the disadvantage of increasing without bound (to $+\infty$ as $\alpha \to +\infty$ and to $-\infty$ as $\alpha \to -\infty$), i.e. it is secular. Despite this, we choose

$$\mathbf{A} = \mathbf{A}_1 = -H_n(\alpha)\,\nabla\Gamma, \tag{5.4}$$

and in order to have a physically acceptable $\mathbf{V}$ in (5.1), we include in ϕ a term which removes the secularity due to $\mathbf{A}$. Thus we write

$$\phi = \phi_1 + \phi^w, \quad \phi^w = \Gamma(\psi)\,(\alpha/s - \tfrac{1}{2}), \tag{5.5}$$

whence
$$\mathbf{V} = \nabla\phi_1 + \frac{\Gamma}{s}\nabla\alpha + \nabla\Gamma\left[\left(\frac{\alpha}{s} - \frac{1}{2}\right) - H_n(\alpha)\right]. \tag{5.6}$$

Notice that $\Gamma\nabla\alpha$ is independent of y, while the term in brackets is the periodic 'saw-tooth' function, with zero average over y (or α).

We are interested here in only the y-dependent part of $\mathbf{V}$, i.e. $\mathbf{v}$. Thus we define

$$\tilde{\phi} = \phi_1 - \overline{\phi}_1 \tag{5.7}$$

and subtract the mean of (5.6) from (5.6) itself, leaving

$$\mathbf{v} = \nabla\tilde{\phi} + \nabla\Gamma(\alpha_n/s),$$
$$\alpha_n/s \equiv (\alpha/s - \tfrac{1}{2}) - H_n(\alpha). \tag{5.8}$$

Note that
$$\nabla\left(\frac{\alpha_n}{s}\right) = \frac{\nabla\alpha}{s}\left[1 - s\sum_{n=-\infty}^{\infty}\delta(\alpha - ns)\right]. \tag{5.9}$$

Finally, using $\operatorname{div}\mathbf{v} = 0$, we obtain an equation for $\tilde{\phi}$ which can be solved for given loading:
$$\nabla^2\tilde{\phi} = -\nabla\Gamma\,.\,\nabla\alpha_n/s - \nabla^2\Gamma\,\alpha_n/s. \tag{5.10}$$

Note the entire right-hand side of (5.10) is $O(\epsilon)$ and thus $\tilde{\phi}$ is also $O(\epsilon)$, as required.

Before proceeding with the approximate solution of (5.10), we must determine $\alpha(x,y,z)$, or $f(x,z)$, since this quantity will appear on both sides of the equation for $\tilde{\phi}$. To accomplish this we recall that the y component of (3.12) determines $f(x,z)$ for given loading. In fact, with the help of (3.2) we have simply

$$\overline{V}_x\frac{\partial f}{\partial x} + \overline{V}_z\frac{\partial f}{\partial z} = \frac{\overline{\Gamma}}{s} + \frac{\delta\Gamma(\psi)}{s} - U. \tag{5.11}$$

This equation can be integrated immediately by the method of characteristics, where here the characteristics are simply the streamlines of the mean flow, i.e. lines along which ψ = constant. Therefore, if τ is the time taken for a fluid particle to drift along a streamline from $x = 0$,

$$f = [\overline{\Gamma}/s + \delta\Gamma(\psi)/s - U]\,\tau. \tag{5.12}$$

The most useful form of the result of the integration† can be written as

$$\frac{\partial f}{\partial x} = \frac{\overline{\Gamma}/s - U}{V_{-\infty}} + \left[1 + \left(\frac{\overline{\Gamma}/s - U}{V_{-\infty}}\right)^2\right]\frac{\delta\Gamma}{V_{-\infty}s}$$
$$- \left(\frac{\overline{\Gamma}/s - U}{V_{-\infty}}\right)^2 \sum_{n=1}^{\infty}\frac{\Gamma_n}{2sV_{-\infty}}\exp(-n\pi x/l)\cos\frac{n\pi z_0(\psi)}{l} + O(\delta\Gamma)^2, \tag{5.13}$$

† The drift time $\tau \equiv \int_0^x dx/\overline{V}_x$, where $\overline{V}_x$ is given by (4.16). We have used the condition $f = 0$ at $x = 0$, which follows from the definition (3.13) of α.

$$\frac{\partial f}{\partial z} = \left[1+\left(\frac{\overline{\Gamma}/s-U}{V_{-\infty}}\right)^2\right]\frac{x}{sU}\frac{d\delta\Gamma}{dz}$$
$$-\left(\frac{\overline{\Gamma}/s-U}{V_{-\infty}}\right)^2 \sum_{n=1}^{\infty}\frac{\Gamma_n}{2sU}(\exp[-n\pi x/l]-1)\sin\frac{n\pi z_0}{l}+O(\delta\Gamma)^2, \quad (5.14)$$

where $z_0(\psi)$ is the value of z at which any line ψ = constant crosses the plane $x = 0$. Since the surfaces α = constant = $y-f(x,z)$ are those in which the trailing vortices lie, we see immediately that the dominant term in (5.13) implies that the vortices leave the blades at essentially the spanwise-averaged outlet angle $\overline{\alpha}_2$. In fact, the leading term in (5.13) is just $\tan\overline{\alpha}_2$. There is, however, some small curvature. More important, although $\partial f/\partial z$ is small in $\delta\Gamma$, it contains a secular term in x, which implies that the wakes became highly distorted far downstream from the blade row. (Note, however, that $\partial f/\partial z = 0$ at $z = 0, l$ for all x.)

While this wake behaviour is quite interesting, and may have important implications far from the blade row, it turns out to be of little consequence near the blade row (see also § 6), which is our main concern in this paper. We therefore leave the question of the effects of the distortion of the wakes to future study.

To determine the induced velocities at the blades (or lifting lines) we must solve (5.10) in the neighbourhood of $x = 0$. This means that we can ignore (in this region) the effect of the small secular term introduced by $\partial f/\partial z$. Moreover, consistent with our treatment of the mean flow, we write Γ in the form [cf. (4.11)]

$$\Gamma = \overline{\Gamma}+\sum_{n=1}^{\infty}\Gamma_n\cos\frac{n\pi z}{l}+O(\delta\Gamma)^2 \quad (5.15)$$

and insert this in the right-hand side of (5.10). Then

$$\nabla\Gamma = \mathbf{k}\left[-\sum_{n=1}^{\infty}\Gamma_n\left(\frac{n\pi}{l}\right)\sin\frac{n\pi z}{l}\right]+O(\delta\Gamma)^2, \quad (5.16a)$$

$$\nabla^2\Gamma = -\sum_{n=1}^{\infty}\Gamma_n\left(\frac{n\pi}{l}\right)^2\cos\frac{n\pi z}{l}+O(\delta\Gamma)^2, \quad (5.16b)$$

and the right-hand side of (5.10) simplifies greatly in the approximation (5.15), at least for small to moderate x (up to a few times l). In this region, the term in $\nabla\Gamma$ in (5.10) drops out altogether (the term $\nabla\Gamma.\nabla\alpha$ is $O(\delta\Gamma)^2$ for moderate x). Similar simplifications occur on the left-hand side of (5.10) when one keeps in mind the fact that $\tilde{\phi}$ is itself $O(\delta\Gamma)$.

We use the Fourier expansion of the saw-tooth function α_n/s, namely

$$\frac{\alpha_n}{s} = \sum_{m=1}^{\infty}\frac{(-2)}{2\pi m}\sin\frac{2\pi m\alpha}{s}$$
$$= \sum_{m=-\infty}^{\infty}{}' \frac{(-1)}{2\pi i m}\exp(2\pi i m\alpha/s), \quad (5.17)$$

where in the last expression the prime on the sum means 'omit $m = 0$', and the real part is implied. This suggests that we seek a solution of (5.10) for small x/l of the form

$$\tilde{\phi} = \sum_{m=-\infty}^{\infty}{}' R_m(z)\exp(2\pi i m\alpha/s)+\hat{\phi}, \quad (5.18)$$

where again the real part is implied and the prime means 'omit $m = 0$', and we define $\hat{\phi}$ by

$$\nabla^2\hat{\phi} = 0, \quad \partial\hat{\phi}/\partial z = 0 \quad \text{at} \quad z = 0, l. \tag{5.19}$$

$\hat{\phi}$ will be of exactly the same form as ϕ^u in (4.4) (with $m = 0$ omitted), except that there will be a minus sign in the argument of the exponential, and of course the coefficients will be different.

In view of (5.18), we have

$$\left.\frac{\partial\tilde{\phi}}{\partial z}\right|_{0,l} = \sum_{m=-\infty}^{\infty}{}' \left[R'_m(z) + \frac{2\pi i m}{s} R_m \frac{\partial\alpha}{\partial z}\right] \exp\left(2\pi i m\alpha/s\right)\bigg|_{0,l} + \left.\frac{\partial\hat{\phi}}{\partial z}\right|_{0,l}, \tag{5.20}$$

in which the second term in the brackets vanishes because

$$[\partial\alpha/\partial z]_{0,l} = -[\partial f/\partial z]_{0,l} = 0$$

[see (5.14)]. Since we also require $[\partial\hat{\phi}/\partial z]_{0,l} = 0$ from (5.19) we obtain the following boundary conditions for the R_m's:

$$\left.\frac{dR_m}{dz}\right|_{0,l} \equiv R'_m\begin{pmatrix}0\\l\end{pmatrix} = 0 \tag{5.21}$$

for each m.

Inserting (5.18), (5.19) and (5.15) in (5.10), we obtain in the usual way for each $R_m(z)$

$$R''_m(z) - \left(\frac{2\pi m}{s}\right)^2 R_m\left[1 + \left(\frac{\overline{\Gamma}/s - U}{V_{-\infty}}\right)^2\right] = \frac{1}{2\pi i m}\frac{d^2\Gamma}{dz^2}, \tag{5.22}$$

which together with (5.21) can be solved by the method of variation of parameters.

It is convenient to introduce

$$\chi_m \equiv 2\pi i m R_m \tag{5.23}$$

and

$$s' \equiv s\left[1 + \left(\frac{\overline{\Gamma}/s - U}{V_{-\infty}}\right)^2\right]^{-\frac{1}{2}} = s\cos\bar{\alpha}_2, \tag{5.24}$$

whereupon (5.22) reduces to

$$\chi''_m - (2\pi m/s')^2\chi_m = \Gamma''(z). \tag{5.25}$$

Upon comparing (5.25) and (6.16), we see that the χ_m's used in this section and the ψ_m's appearing in the Trefftz-plane calculation (§6) are related simply by

$$\psi'_m(z) = (2/s')\chi_m. \tag{5.26}$$

The solution of (5.25) is $\left[\text{recalling that } \chi'_m\begin{pmatrix}0\\l\end{pmatrix} = 0 \text{ and } \Gamma'\begin{pmatrix}0\\l\end{pmatrix} = 0\right]$

$$\chi_m = \frac{1}{\sinh(2\pi m l/s')}\left\{\cosh\left[\frac{2\pi m}{s'}(l-z)\right]\int_0^z \Gamma'\sinh\left(\frac{2\pi m}{s'}t\right)dt - \cosh\left(\frac{2\pi m}{s'}z\right)\int_z^l \Gamma'\sinh\left[\frac{2\pi m}{s'}(l-t)\right]dt\right\}, \tag{5.27}$$

which can be verified readily by direct substitution in (5.25).

We may now collect our results for the downstream flow field and apply the appropriate matching conditions across the blade row (we recall that the mean flow has already been matched in §4). Thus

$$\mathbf{V}^d = \bar{\mathbf{V}}^d + \mathbf{v}^d = \bar{\mathbf{V}}^d + \nabla\tilde{\phi} + (\alpha_n/s)\nabla\Gamma, \tag{5.28}$$

where
$$\tilde{\phi} = \sum_{m=-\infty}^{\infty}{}' (2\pi i m)^{-1}\chi_m \exp(2\pi i m\alpha/s) + \hat{\phi} \tag{5.29}$$

and
$$\hat{\phi} = \sum_{m=-\infty}^{\infty}{}' \sum_{n=0}^{\infty} A^d_{nm} \exp(-\lambda_{nm}x)\cos\frac{n\pi z}{l}\exp(2\pi i m y/s). \tag{5.30}$$

Once again, the λ_{nm} satisfy (4.5).

The complete matching is accomplished by requiring

$$V^u_x|_{x=0^-} = V^d_x|_{x=0^+}, \tag{5.31}$$

or [cf. (4.13)]
$$u^u|_{x=0^-} = u^d|_{x=0^+} \tag{5.32}$$

and
$$\Delta V_y \equiv V^d_y|_{x=0^+} - V^u_y|_{x=0^-} = \frac{\Gamma}{s}\left(1 + \sum_{m=-\infty}^{\infty}{}' \exp(2\pi i m y/s)\right). \tag{5.33}$$

With (4.18) in mind, this becomes

$$\Delta v = v^d|_{x=0^+} - v^u|_{x=0^-} = \frac{\Gamma}{s}\sum_{m=-\infty}^{\infty}{}' \exp(2\pi i m y/s), \tag{5.34}$$

where the sum in (5.33) [and also in (5.34) and (5.17)] converges only in the sense of 'generalized functions', i.e. it is the Fourier representation of the required periodic delta function. The condition (5.33), of course, represents the jump in V_y imposed by the bound vortex lines at the blade row. Conditions (5.32) and (5.34) together imply $\Delta w = 0$.

For convenience in carrying out the matching, we introduce the Fourier expansion

$$R_m(z) = \frac{1}{2\pi i m}\chi_m(z) = \sum_{n=0}^{\infty} h_{nm}\cos\frac{n\pi z}{l} \tag{5.35}$$

with the usual inverse formula for the coefficients h_{nm}, $n \geqslant 1$. We note, from (5.22) or (5.25), that the h_{nm} are pure imaginary (each $\chi_m(z)$ is real) and, moreover, that

$$h_{0m} = 0 = \frac{1}{l}\int_0^l dz\, R_m(z).$$

Finally, in actually carrying out the calculations required in (5.32) and (5.34), we again make use of the approximation (4.9), expressed explicitly in the form (5.15). The result determines the remaining unknown coefficients,

$$A^d_{nm} = \frac{\Gamma_n}{4\pi i m} - \frac{h_{nm}}{2}\left[1 + \frac{\bar{\Gamma}/s - U}{V_{-\infty}}\frac{2\pi i m}{s\lambda_{nm}}\right], \tag{5.36}$$

$$A^u_{nm} = -\frac{\Gamma_n}{4\pi i m} + \frac{h_{nm}}{2}\left[1 - \frac{\bar{\Gamma}/s - U}{V_{-\infty}}\frac{2\pi i m}{s\lambda_{nm}}\right], \tag{5.37}$$

and thereby the entire upstream and downstream flow field, described by (5.28)–(5.30) and (4.4) [see also (4.12) and (4.15)]. Notice that the $A^{u,d}_{0m}$ are particularly simple, since $h_{0m} = 0$ and $\Gamma_0 \equiv \bar{\Gamma}$.

The induced velocities at the lifting lines $(0,0,z)$ can now be computed, yielding

$$\left.\begin{aligned} \langle v\rangle &\equiv \tfrac{1}{2}[v^u(0,0,z)+v^d(0,0,z)] \\ &= \frac{1}{2}\frac{\Gamma}{s}+\frac{1}{2}\sum_{m=-\infty}^{\infty}{}' \frac{\chi_m(z)}{s}, \\ \langle u\rangle &= \tfrac{1}{2}[\bar{V}_x(0,z)-V_{-\infty}]-\tan\bar{\alpha}_2 \sum_{m=-\infty}^{\infty}{}' \frac{\chi_m(z)}{s}. \end{aligned}\right\} \tag{5.38}$$

$\bar{V}_x$ is given by (4.16), while $\langle u\rangle$ is defined in the same way as $\langle v\rangle$, and purely imaginary terms have been omitted. Since the induced velocity perpendicular to the mean outlet angle is $\langle v_y'\rangle \equiv -\langle u\rangle\sin\bar{\alpha}_2+\langle v\rangle\cos\bar{\alpha}_2$, equations (5.38) enable us to compute the velocity normal to the outlet flow at each lifting line:

$$\begin{aligned} \langle v_y'\rangle &= \frac{1}{2}\left\{\frac{\bar{\Gamma}\cos^2\bar{\alpha}_2}{s'}+\frac{\Gamma-\bar{\Gamma}}{s'}(\cos^2\bar{\alpha}_2+\sin^2\bar{\alpha}_2)+\sum_{m=1}^{\infty}\frac{2\chi_m(z)}{s'}(\cos^2\bar{\alpha}_2+\sin^2\bar{\alpha}_2)\right\} \\ &= \frac{1}{2}\left\{\frac{\bar{\Gamma}\cos^2\bar{\alpha}_2}{s'}+\frac{\Gamma-\bar{\Gamma}}{s'}+\sum_{m=1}^{\infty}\psi_m'(z)\right\}, \end{aligned} \tag{5.39}$$

where we have used (5.26). The first term in (5.39) is due to the bound vortices and would not normally be included in a calculation of the 'downwash velocity'. The results of this section are verified in § 6, using a Trefftz-plane approach.

Since the requirement $\delta\Gamma/\bar{\Gamma} \ll 1$ can be satisfied for larger $\delta\Gamma$'s (and hence larger $d\Gamma/dr$) as $\bar{\Gamma}$ is increased, the induced velocity due to the shed vorticity [the third term in (5.39)] can increase significantly with the loading $\bar{\Gamma}$. The effect is zero, however, if $\Gamma = \bar{\Gamma} =$ constant. Numerical calculations of the associated deviation angles are deferred here, as a practical matter, for the annular case. Useful examples have been worked out numerically for that case by Cheng (1975), showing that three-dimensional (blade-to-blade) effects are indeed significant, especially for moderate numbers of blades (10–20, say), and especially near the hub and tip regions of the blading.

6. Trefftz-plane approximation and its justification

In the earlier sections of this paper the suggestion was made that a useful approximation to the flow downstream of a blade row can be obtained by describing the trailing vorticity as being convected by the tangential mean of the actual flow field. When the strength of the trailing vorticity is weak, $O(\epsilon)$, this leads further to the result that the blade-to-blade variations are also $O(\epsilon)$, while the mean flow is $O(1)$ (§ 4). In addition, this model yields the rather satisfactory result that the trailing vorticity leaves the blades at approximately the mean outlet angle, twisting up upon itself (§ 5) only much later, owing to secular terms which are strictly $O(\epsilon^2)$. Thus a principal feature of the theory is that the nature of the velocities induced at the blades by the trailing vorticity is associated with the wakes leaving the blades at this outlet angle, and the twisting up of the wakes has a negligible effect on the velocities induced at the blades.

However, as can be seen from § 4, the mean flow itself can, if we wish, be further separated into an irrotational part and a rotational part, which is, in

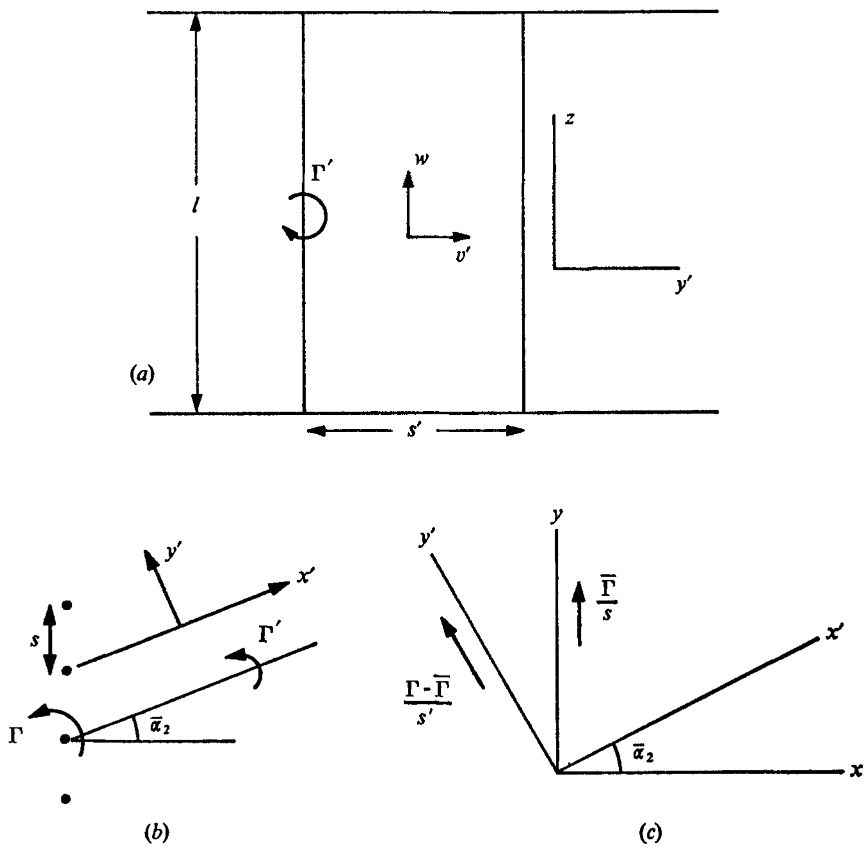

FIGURE 2. (*a*), (*b*) Notation used in Trefftz-plane analysis. (*c*) Mean velocities induced by shed and bound vortices.

fact, $O(\epsilon)$. A division of the flow in this manner is more in line with the separation of flow components suggested by Hawthorne (1967) for problems involving "small shear, large disturbance". In this categorization of flow, the primary flow turns out to be purely irrotational and convects the vortex filaments, which in turn induce the perturbation velocities in which we are interested.

When this picture is adopted, in which all terms $O(\epsilon^2)$ are consistently neglected, the convecting flow, which in fact is two-dimensional, does not twist up [the wakes do not distort to $O(\epsilon)$] and, as a result, a Trefftz-plane analysis becomes justified. Here, the flow far downstream is pictured as being the result of convection of the wakes along surfaces at an angle $\bar{\alpha}_2$.

The notation for the corresponding Trefftz-plane is shown in figures 2(*a*) and (*b*). As the Trefftz plane is at 'infinity' far downstream, derivatives of the velocities in the x' direction vanish, so that the continuity equation becomes

$$\partial v'/\partial y' + \partial w/\partial z = 0 \tag{6.1}$$

and we may introduce a stream function $\psi(y', z)$, where

$$v' = \partial\psi/\partial z, \quad w = -\partial\psi/\partial y'. \tag{6.2}$$

Here v' and w are the velocities relative to the blades induced by the shed vortex sheets a distance $s' = s\cos\bar{\alpha}_2$ apart, where s is the blade spacing, and the orientation of (v', y') relative to (v, y), etc., as used in § 5, is illustrated in figure 2. Using (6.2) we may write the vorticity as

$$\frac{\partial w}{\partial y'} - \frac{\partial v'}{\partial z} = -\left(\frac{\partial^2 \psi}{\partial y'^2} + \frac{\partial^2 \psi}{\partial z^2}\right)$$

$$= -\Gamma'(z) \sum_{n=-\infty}^{\infty} \delta(y' - ns'). \tag{6.3}$$

On writing the delta functions representing the vortex sheets in series form (cf. § 5) we obtain

$$\psi_{y'y'} + \psi_{zz} = \frac{\Gamma'}{s'}\left(1 + 2\sum_{m=1}^{\infty} \cos m2\pi y'/s'\right). \tag{6.4}$$

The solution may be obtained by writing

$$\psi = \sum_{m=0}^{\infty} \psi_m(z) \cos m2\pi y'/s' \tag{6.5}$$

with boundary conditions $\psi_m(0) = \psi_m(l) = 0$, giving $w = 0$ at the walls, $z = 0$ and l. For $m = 0$

$$\psi_0 = \int_0^z \frac{\Gamma - \bar{\Gamma}}{s'} dz, \tag{6.6}$$

which satisfies the boundary conditions. For $m \neq 0$

$$\psi_m'' - (m2\pi/s')^2 \psi_m = 2\Gamma'/s', \tag{6.7}$$

solutions of which may be obtained, as before, by the method of variation of parameters. At $y' = 0$ we obtain

$$v'(\infty, 0, z) = \psi_z(0, z) = \sum_{m=0}^{\infty} \psi_m'$$

$$= \frac{\Gamma - \bar{\Gamma}}{s\cos\bar{\alpha}_2} + \sum_{m=1}^{\infty} \psi_m'. \tag{6.8}$$

To this we must add the primary flow, which can be regarded as the sum of the upstream flow, the blade speed and the contributions from the bound vortices. Now bound vortices of strength $\Gamma(z)$ which vary periodically in the z direction induce at infinity a velocity $\bar{\Gamma}/s$ in the y direction (see figure 2c), because the effect of the periodically variable component averages out. Figure 2(c) shows the mean velocities produced by the bound and shed vortices (the term $m = 0$). Resolving these in the y direction we obtain

$$\bar{V}_{y\infty} = \frac{\bar{\Gamma}}{s} + \frac{\Gamma - \bar{\Gamma}}{s\cos\bar{\alpha}_2}\cos\bar{\alpha}_2 = \Gamma/s,$$

which agrees with the result obtained from consideration of the local circulation about each blade in the cascade.

In the x direction we obtain

$$\bar{V}_{x\infty} - V_{-\infty} = (\bar{\Gamma} - \Gamma)\, s^{-1} \tan \bar{\alpha}_2,$$

which is the change in axial velocity determined from the actuator-disk solution (4.17).

The velocities due to the vortices at the blade row are half those obtained in the Trefftz plane, and, specifically, the upwash is given by

$$\langle v_y' \rangle \equiv v'(0,0,z) = \frac{1}{2}\left\{\Gamma - \bar{\Gamma} + \bar{\Gamma}\cos^2\bar{\alpha}_2 + s\cos\bar{\alpha}_2 \sum_{m=1}^{\infty} \psi_m'\right\}\Big/ s\cos\bar{\alpha}_2. \quad (6.9)$$

The first two terms are obtained from actuator-disk theory. The third term is precisely the same as the term obtained in (5.39).

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REFERENCES

BRAGG, S. L. & HAWTHORNE, W. R. 1950 Some exact solutions of the flow through annular cascade actuator discs. *J. Aero. Sci.* **17**, 243.

CHENG, W. K. 1975 A three dimensional theory for the velocity induced by a heavily loaded annular cascade of blades. S.M. thesis, Department of Aeronautics and Astronautics, M.I.T.

FALCÃO, A. F. DE O. 1970 Three-dimensional flow analysis in axial turbomachines. Ph.D. thesis, Engineering Department, Cambridge University.

HAWTHORNE, W. R. 1967 The applicability of secondary flow analyses to the solution of lateral flow problems. *Fluid Mechanics of Internal Flow*, pp. 238–265. Elsevier.

LIGHTHILL, M. J. 1958 *An Introduction to Fourier Analysis and Generalised Functions.* Cambridge University Press.

MCCUNE, J. E. 1958*a* A three-dimensional theory of axial compressor blade rows – subsonic, supersonic and transonic. Ph.D. thesis, Cornell University. (See also US AFOSR TN-58-72.)

MCCUNE, J. E. 1958*b* A three-dimensional theory of axial compressor blade rows – application in subsonic and supersonic flows. *J. Aero. Space Sci.* **25**, 544.

MCCUNE, J. E. 1958*c* The transonic flow field of an axial compressor blade row. *J. Aero. Space Sci.* **25**, 616.

MCCUNE, J. E. & DHARWADKAR, S. P. 1972 Lifting line theory for subsonic axial compressor rotors. *M.I.T. Gas Turbine Lab. Rep.* no. 110.

MCCUNE, J. E. & OKUROUNMU, O. 1974 Three-dimensional flow in transonic axial compressor blade rows. *Proc. Int. Symp. Fluid Mech., Acoustics & Design of Turbomachinery, Pennsylvania State University*, 1970 (ed. B. Lakshminarayana, W. R. Britisch & W. S. Gearhart), *N.A.S.A. Special Publ.* SP-304-Pt-1 (N75-11174), p. 155.

MORTON, K. B. 1974 Three-dimensional, compressible flow through highly-loaded, rectilinear cascade. S.M. thesis, Department of Aeronautics and Astronautics, M.I.T.

NAMBA, M. 1974 Lifting surface theory for a rotating subsonic or transonic blade row. *Aero. Res. Counc. R. & M.* no. 3740.

OATES, G. C. 1971 Actuator disc theory for incompressible highly rotating flows. *A.S.M.E. Paper*, 71-WA/FE-18.

OKUROUNMU, O. 1967 Wave drag in transonic axial compressors. Sc.D. thesis, Mechanical Engineering Department, M.I.T.

OKUROUNMU, O. & MCCUNE, J. E. 1970 Three-dimensional vortex theory of axial compressor blade rows at subsonic and transonic speeds. *A.I.A.A. J.* **8**, 1275.

OKUROUNMU, O. & MCCUNE, J. E. 1971 Transonic lifting surface theory for axial flow compressors. *United Aircraft Res. Labs., E. Hartford, Ct, Rep.* K 213580-1.

OKUROUNMU, O. & MCCUNE, J. E. 1974*a* The lifting surface theory of axial compressor blade rows. Part I: the subsonic compressor. *A.I.A.A. J.* **12**, 1363.

OKUROUNMU, O. & MCCUNE, J. E. 1974*b* The lifting surface theory of axial compressor blade rows. Part II. The transonic compressor. *A.I.A.A. J.* **12**, 1372.

REISSNER, H. 1937 On the vortex theory of the screw propeller. *J. Aero. Sci.* **5**, 1.

THWAITES, B. (ed.) 1960 *Incompressible Aerodynamics*. Oxford: Clarendon Press.